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The crossing numbers of generalized Petersen graphs with small order[☆]

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Abstract

The generalized Petersen graph $P(n, k)$ is an undirected graph on $2n$ vertices with $V(P(n, k)) = \{a_i, b_i : 0 \le i \le n-1\}$ and $E(P(n, k)) = \{a_i b_i, a_i a_{i+1}, b_i b_{i+k} : 0 \le i \le n-1$, subscripts modulo $n\}$. Fiorini claimed to have determined the crossing numbers of $P(n, 3)$ and showed all the values of $cr(P(n, k))$ for n up to 14, except 12 unknown values. Lovrečič Saražin proved $cr(P(10, 4)) = cr(P(10, 6)) = 4$. Richter and Salazar found a gap in Fiorini's paper, which invalidated his principal results about $cr(P(n, 3))$, and gave the correct proof for $cr(P(n, 3))$. In this paper, we show the crossing numbers of all $P(n, k)$ for n up to 16.

Keywords: Generalized Petersen graph; Planar graph; Crossing number; Embedding

1. Introduction

We consider only finite undirected graphs without loops or multiple edges.

A *graph* $G = (V, E)$ is a set V of vertices and a subset E of the unordered pairs of vertices, called edges. Let $p = |V|$ and $q = |E|$.

The *generalized Petersen graph* $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{a_i, b_i : 0 \le i \le n-1\}$ and $E(P(n, k)) = \{a_i b_i, a_i a_{i+1}, b_i b_{i+k} : 0 \le i \le n-1$, subscripts modulo $n\}$.

A *drawing* is a mapping of a graph into a surface. The vertices go into distinct points called nodes. An edge and its incident vertices map into a homeomorphic image of the closed interval [0, 1] with the relevant nodes as endpoints and the interior, an arc, containing no node. A drawing is *good* if it satisfies (i) no two arcs incident with a common node have a common point; (ii) no two arcs have more than one point in common; (iii) no arc has a self-intersection; and (iv) no three arcs have a point in common other than a node. A common point of two arcs is a crossing. An *optimal drawing* in a given surface is a good drawing which exhibits the least possible number of crossings. This number is the *crossing number* of the graph for the surface. We denote the crossing number of G for the Euclidean plane (or sphere) by $cr(G)$, a good drawing of G in a given surface by $D(G)$, and the number of crossings of this drawing by $\nu_{D(G)}$.

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Table 1
Fiorini's crossing numbers of $P(n, k)$ for $n \leq 14$.

k	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1				0	0	0	0	0	0	0	0	0	0	0	0
2				0	0	2	0	3	0	3	0	3	0	3	0
3					0	2	1	3	4	2	4*	5	4	5*	6
4						0	0	3	1	2*	?	5	?	5*	?
5							0	3	4	3	1	2*	?	?	6
6								0	0	2	?	2*	1	3	?
7									0	3	4*	5	?	3	1
8										0	0	5	?	?	?
9											0	3	4	5*	6
10												0	0	5*	?
11													0	3	6
12														0	0
13															0

A surface means a plane in this paper. It is clear that for any good drawing $D(G)$ of G in the plane,

$$cr(G) \leq \nu_{D(G)}.$$

Calculating the crossing number of a given graph is, in general, an elusive problem. As Garey and Johnson have proved, the problem of determining the crossing number of an arbitrary graph is NP-complete [4]. So far, the crossing numbers of very few families of graphs are known exactly.

In 1986, Fiorini [2] claimed to have determined the crossing numbers of certain families of generalized Petersen graphs. He proved

(1) $cr(P(3h, 3)) = h, h \geq 4$,
(2) $h + 3 \geq cr(P(3h + 1, 3)) \geq h + 1, h \geq 3$,
(3) $cr(P(3h + 2, 3)) = h + 2, h \geq 2$.

He also showed values of $cr(P(n, k))$ for n up to 14 (see Table 1, where we mark the unknown number with ? and the incorrect number with *).

Dan Mcquillan and Richter [7] showed that $cr(P(10, 3)) \geq 5$.

Later, Richter and Salazar [8] further pointed out that Fiorini's paper contained a gap which invalidated the principal results. They proved that

(1) $cr(P(3h, 3)) = h, h \geq 4$,
(2) $cr(P(3h + 1, 3)) = h + 3, h \geq 3$,
(3) $cr(P(3h + 2, 3)) = h + 2, h \geq 3$.

Lovrečič Saražin [6] proved $cr(P(10, 4)) = cr(P(10, 6)) = 4$.

In 2003, Fiorini and Gauci [3] proved that $cr(P(3k, k)) = k$ for $k \geq 4$.

In [9] Watkins showed

(1) $P(n, k) \cong P(n, n - k)$;
(2) If $1 \leq k, m \leq n - 1$ and $km \equiv 1 \pmod{n}$, then $P(n, m) \cong P(n, k)$.

We examine the crossing numbers of generalized Petersen graphs for n up to 16, fill up the missing numbers in Table 1, and show the new values of $cr(P(n, k))$ for $n \leq 16$ in Table 2. Since $P(n, k) \cong P(n, n - k)$, we only list the values of $cr(P(n, k))$ for $k \leq n/2$. Among these, three values

$$cr(P(10, 3)) = 6, \quad cr(P(11, 3)) = 5, \quad cr(P(12, 3)) = 4$$

were obtained while the second author visited professor R. Bruce in 1996, and were used by him as the bases of induction in the proof for $cr(P(n, 3))$ in his paper [8].

Table 2
The crossing numbers of $P(n, k)$ for $n \leq 16$.

k	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1				0	0	0	0	0	0	0	0	0	0	0	0	0	0
2					0	2	0	3	0	3	0	3	0	3	0	3	0
3							1	3	4	2	6	5	4	7	6	5	8
4									1	3	4	5	4	7	8	10	8
5											1	3	8	9	6	5	8
6													1	3	7	10	12
7															1	3	9
8																	1

2. Some structural lemmas

By [1,5,6], we have Lemma 2.1.

Lemma 2.1. (1) $cr(P(n, 1)) = 0$;
(2) $cr(P(2h, 2)) = 0$, $h \geq 2$; $cr(P(5, 2)) = 2$; $cr(P(2h + 1, 2)) = 3$, $h \geq 3$;
(3) $cr(P(2h, h)) = 1$, $h \geq 3$;
(4) $cr(P(10, 4)) = 4$.

By [8], we have Lemma 2.2.

Lemma 2.2. (1) $cr(P(3h, 3)) = h$, $h \geq 4$;
(2) $cr(P(3h + 1, 3)) = h + 3$, $h \geq 3$;
(3) $cr(P(3h + 2, 3)) = h + 2$, $h \geq 2$.

By [3], we have Lemma 2.3.

Lemma 2.3. $cr(P(3k, k)) = k$, $k \geq 4$.

By [9], we have Lemma 2.4.

Lemma 2.4. (1) $P(n, k) \cong P(n, n - k)$;
(2) *If* $1 \leq k, m \leq n - 1$ *and* $km \equiv 1 \pmod{n}$, *then* $P(n, m) \cong P(n, k)$.

In Fig. 1(a) and (b), we show good drawings of $P(4h + 2, 2h)$ and $P(4h + 2, 4)$ for $h \geq 3$ with $2h + 1$ and $2h + 2$ crossings, respectively. Hence

Lemma 2.5. (1) $cr(P(4h + 2, 2h)) \leq 2h + 1$, $h \geq 3$;
(2) $cr(P(4h + 2, 4)) \leq 2h + 2$, $h \geq 3$.

In all figures, t and $\underline{t}$ stand for a_t and b_t, respectively.
Let $f(n, k)$ denote the numbers in Table 2; then, by Lemmas 2.1–2.5, we have

Lemma 2.6. $cr(P(n, k)) \leq f(n, k)$ *for* $n \leq 16$.

Proof. If $k \leq 3$ or $n = 2k$ or $(n, k) = (10, 4)$, by Lemmas 2.1 and 2.2, we have $cr(P(n, k)) \leq f(n, k)$. By Lemmas 2.2–2.5, we have

$$cr(P(9, 4)) = cr(P(9, 5)) = cr(P(9, 2)) = 3;$$
$$cr(P(11, 5)) = cr(P(11, 6)) = cr(P(11, 2)) = 3;$$
$$cr(P(13, 6)) = cr(P(13, 7)) = cr(P(13, 2)) = 3;$$
$$cr(P(15, 7)) = cr(P(15, 8)) = cr(P(15, 2)) = 3;$$
$$cr(P(11, 4)) = cr(P(11, 3)) = 5;$$
$$cr(P(13, 4)) = cr(P(13, 10)) = cr(P(13, 3)) = 7;$$

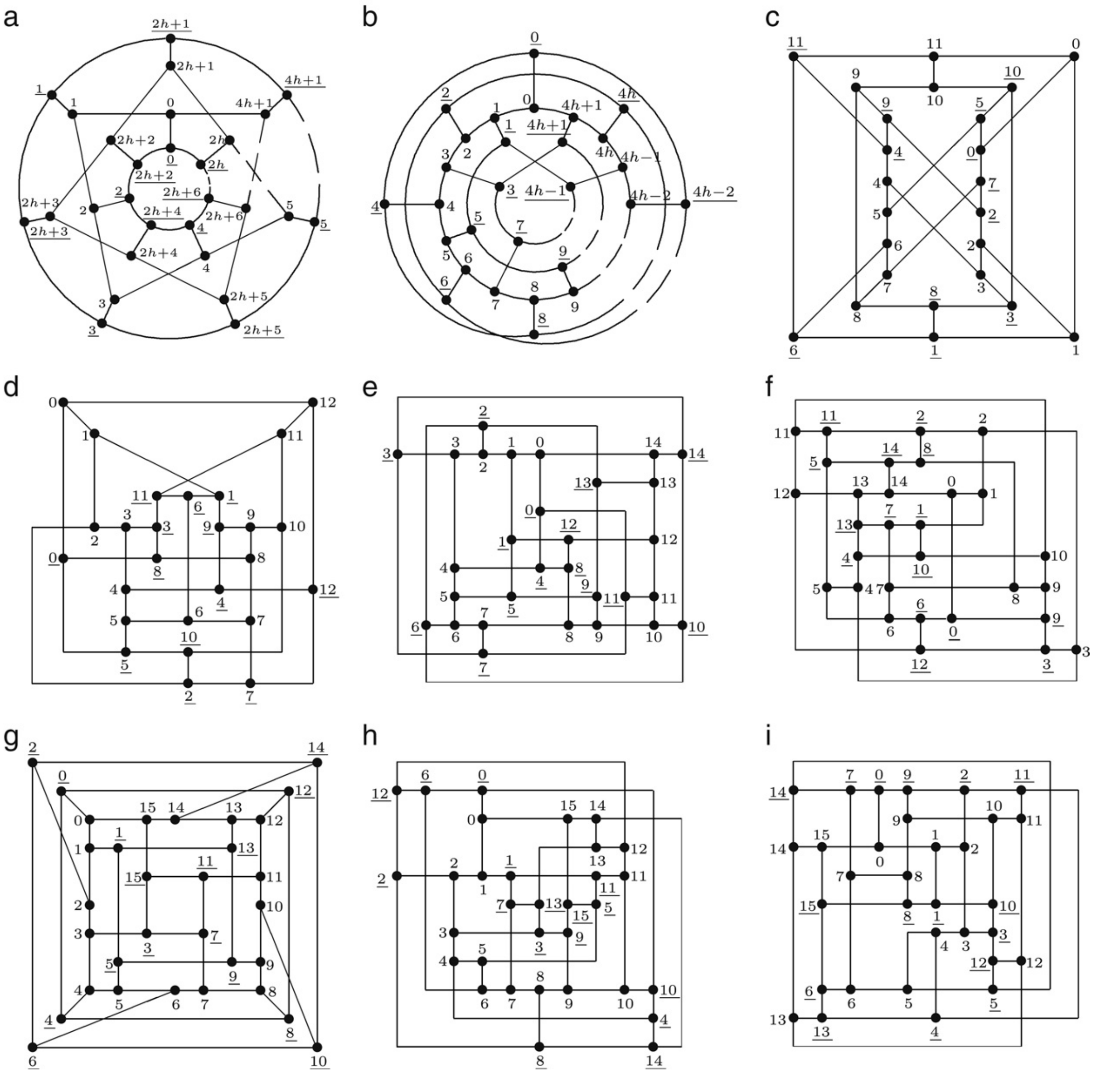

Fig. 1. Some good drawings of $P(n,k)$: (a) $cr(P(4h+2,2h)) \le 2h+1$, (b) $cr(P(4h+2,4)) \le 2h+2$, (c) $cr(P(12,5)) \le 8$, (d) $cr(P(13,5)) \le 9$, (e) $cr(P(15,4)) \le 10$, (f) $cr(P(15,6)) \le 10$, (g) $cr(P(16,4)) \le 8$, (h) $cr(P(16,6)) \le 12$, and (i) $cr(P(16,7)) \le 9$.

$$cr(P(14,5)) = cr(P(14,3)) = 6;$$
$$cr(P(16,5)) = cr(P(16,11)) = cr(P(16,3)) = 8;$$
$$cr(P(12,4)) = 4; \quad cr(P(15,5)) = 5;$$
$$cr(P(14,4)) \le 8; \quad cr(P(14,6)) \le 7.$$

In Fig. 1(c–i), we show good drawings of $P(n,k)$ with $f(n,k)$ crossings for $(n,k) \in \{(12,5), (13,5), (15,4), (15,6), (16,4), (16,6), (16,7)\}$. Hence

$$cr(P(n,k)) \le f(n,k), \quad (n,k) \in \{(12,5), (13,5), (15,4), (15,6), (16,4), (16,6), (16,7)\}. \quad \square$$

3. The crossing numbers of $P(n, k)$

In this section, we introduce Algorithm CCN (Calculate Crossing Number) which uses a branch and bound method to calculate the crossing numbers of $P(n, k)$ for $n \leq 16$ and $k \leq n/2$, to prove that $cr(P(n, k)) = f(n, k)$.

A graph is said to be *embeddable* in a surface, if it can be drawn in the surface so that its edges intersect only at their ends. Such a drawing is called an *embedding* in the surface of G.

A graph is called a *planar graph* if it can be embedded in a plane, otherwise it is called a *non-planar graph*.

Let D be a good drawing of a non-planar graph G. We refer to d_m as a *sub-drawing* of D obtained by removing m edges $\{e_1, e_2, \dots, e_m\} \subseteq E$ from D. If there is no crossing in d_m, we refer to it as a *planar sub-drawing* of D.

By [2], we have Lemma 3.1.

Lemma 3.1. *If $S_{m_0} = \{e_1, e_2, \dots, e_{m_0}\} \subset E$ is the minimum edge subset of a graph G such that $H_{m_0} = G - S_{m_0}$ is a planar subgraph of G, then $cr(G) \geq m_0$.*

Let

$$P_m = \{p_m : p_m \text{ is a planar subgraph of } G \text{ obtained by deleting } m \text{ edges from } G\};$$
$$D_m = \{d_m : d_m \text{ is an embedding of } p_m \in P_m\}.$$

Algorithm CCN
Procedure CCN;
Begin
0. If (G is a *planar graph*) Then $cr_1 = 0$
  Else
  Begin
1. $m = 0$; $cr_1 = \infty$; $S_0 = \{G\}$;
    Repeat
2.   Repeat
       $m = m + 1$; $S_m = \emptyset$; $P_m = \emptyset$; $D_m = \emptyset$;
       For every $g_{m-1} \in S_{m-1}$ Do
         For every edge $e \in E(g_{m-1})$ Do
         Begin
           $g_m = g_{m-1} - e$
           $S_m = S_m \cup \{g_m\}$
           If g_m is a *planar graph* Then $P_m = P_m \cup \{g_m\}$;
         End
     Until $P_m \neq \emptyset$;
3.   For every $p_m \in P_m$ Do
       For every *embedding* d_m of p_m Do
         $D_m = D_m \cup \{d_m\}$;
4.   For every $d'_0 = d_m \in D_m$ Do
     Begin
       Denote $E(G) - E(d'_0)$ by $\{e_1, e_2, \dots, e_m\}$;
       $D'_0 = \{d'_0\}$;
       For $j = 1$ to m Do
       Begin
         $D'_j = \emptyset$;
         For every *drawing* $d'_{j-1} \in D'_{j-1}$ Do
         Begin
           Let D_{e_j} be the set of all possible drawings obtained
             by putting e_j back into d'_{j-1};
           For every $d_{e_j} \in D_{e_j}$ Do
             If $\nu_{d_{e_j}} \leq cr_1$

If $j = m$ Then $cr_1 = \nu_{d_{e_j}}$
Else $D'_j = D'_j \cup \{d_{e_j}\}$
End
End
End
5. Until $m \geq cr_1$
End
End;

Theorem 3.2. *Algorithm CCN calculates the crossing number of graph G correctly.*

Proof. Algorithm CCN will return with $cr_1 = 0$ in step 0 for a planar graph G.

For any non-planar graph G, there exists an optimal drawing D of G. By deleting some crossed edges from D, we can get planar sub-drawings of D. Let S_D be the set of all planar sub-drawings of D and $d_{m'} \in S_D$ be a planar sub-drawing with m' deleted edges.

Algorithm CCN examines all the possible situations of deleting m edges for $cr_1 \geq cr(G) \geq m' \geq m \geq 1$.

In step 2, the inner repeat-until statement of Algorithm CCN deletes m edges of G in all the possible ways to get P_m.

In step 3, Algorithm CCN constructs all the embeddings for every $p_m \in P_m$ and puts them into the set D_m. Hence we have

$$d_{m'} \in D_{m'}.$$

In step 4, for every drawing $d_m \in D_m$, Algorithm CCN constructs all the drawings where m edges are put back and the numbers of crossings are not greater than the current minimum number of crossings cr_1, including D (where $m = m'$). Thus, it should be that $cr_1 \leq \nu_D = cr(G)$ when Algorithm CCN terminates, i.e., Algorithm CCN calculates the crossing number of G correctly. □

Algorithm CCN uses a branch and bound method to calculate the crossing number of a graph G. It puts back all the deleted edges in all possible drawings to every embedding of planar subgraphs of G. There are two bounding conditions here. Bounding-condition 1: the algorithm restores the remaining deleted edges in all the possible drawings in step 4 only if the number of crossings of the current drawing $\nu_{d_{e_j}}$ is not greater than the current minimal number of crossings cr_1. Since if $\nu_{d_{e_j}} > cr_1$, then, no matter how we put the remaining deleted edges back, we cannot get a drawing of G with the number of crossings smaller than cr_1. Bounding-condition 2: the algorithm only calculates the number of crossings for the embedding of the planar subgraph obtained by deleting at most $m \leq cr_1$ edges from G. Since if $m > cr_1$, putting m edges back will add at least $m > cr_1$ crossings.

Algorithm CCN examines all the drawings of G for $\nu_D \leq cr_1$. By the proof of Theorem 3.2, when Algorithm CCN terminates, $cr_1 = cr(G)$. Thus the algorithm constructs all the optimal drawings of G.

To make Algorithm CCN more efficient, we further modify its step 4 as follows:

4. For every $d'_0 = d_m \in D_m$ Do
Begin
Denote $E(G) - E(d'_0)$ by $\{e_1, e_2, \ldots, e_m\}$
$D'_0 = \{d'_0\}$;
For j = 1 to m Do
Begin
$D'_j = \emptyset$;
For every *drawing* $d'_{j-1} \in D'_{j-1}$ Do
Begin
Let g_{j-1} be the graph corresponding to d'_{j-1};
Let $E(G) - E(g_{j-1})$ be $\{e_{i_j}, e_{i_{j+1}}, \ldots, e_{i_m}\}$;
Let $\nu^*_{e_{i_t}}$ be the minimum increments of the crossings induced by putting e_{i_t} back to d'_{j-1};
$cr_2 = \nu_{d'_{j-1}} + \sum_{t=j}^{m} \nu^*_{e_{i_t}}$;

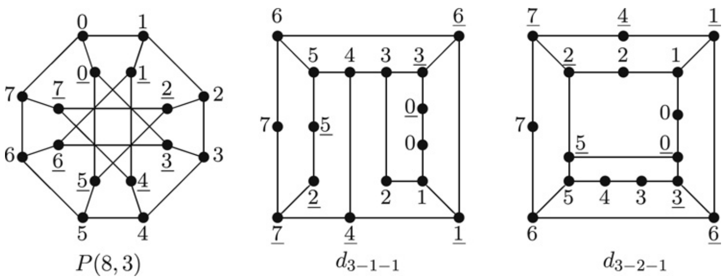

Fig. 2. $P(8, 3)$ and the embeddings of its planar subgraphs.

If $cr_2 < cr_1$ Then
Begin
    $m^* = (cr_1 - cr_2)/(m - j + 1)$;
    For $t = j$ to m Do
    Begin
        Let $D_{e_{i_t}}$ be the set of all the possible drawings obtained by putting e_{i_t} back into d'_{j-1} and with at most $\nu^*_{e_{i_t}} + m^*$ added crossings;
        For every $d_{e_{i_t}} \in D_{e_{i_t}}$ Do
            If $\nu_{d_{e_{i_t}}} < cr_1$ then
                If $j = m$ then $cr_1 = \nu_{d_{e_{i_t}}}$
                Else $D'_j = D'_j \cup \{d_{e_{i_t}}\}$
    End
End
End
End
End

Now the Bounding-condition 1 is replaced by Bounding-conditions 1.1–1.3. Bounding-condition 1.1: the algorithm goes on putting back the remaining deleted edges only if $cr_2 < cr_1$. Since if $cr_2 \geq cr_1$, no matter how the remaining deleted edges were put back, we will get drawings with at least $cr_2(\geq cr_1)$ crossings. Bound-condition 1.2: when putting e_{i_t} back to the drawing d'_{j-1}, we only examine the drawings whose increment of crossings is at most $\nu^*_{e_{i_t}} + (cr_1 - cr_2)/(m - j + 1)$. For if all the increments of the crossings induced by putting the edge e_{i_t} back are greater than $\nu^*_{e_{i_t}} + (cr_1 - cr_2)/(m - j + 1)$, then the number of crossings of the finally constructed drawing of G, $\nu_{D(G)}$, will be at least $\nu_{d'_{j-1}} + \sum_{t=j}^{m} (\nu^*_{e_{i_t}} + (cr_1 - cr_2)/(m - j + 1) + 1) \geq (\nu_{d'_{j-1}} + \sum_{t=j}^{m} \nu^*_{e_{i_t}}) + cr_1 - cr_2 = cr_2 + cr_1 - cr_2 = cr_1$. Bound-condition 1.3: the algorithm restores the remaining deleted edges only if $\nu_{d_{e_{i_t}}} < cr_1$. Since if $\nu_{d_{e_{i_t}}} \geq cr_1$, then no matter how we added the remaining deleted edges, we will get a drawing of G with at least cr_1 crossings.

Further, since we need only to calculate the crossing number of G, or to construct one optimal drawing of G, we also change the Bounding-condition 2 so that the algorithm only calculates the number of crossings for embedding of the planar subgraph obtained by deleting at most $m < cr_1$ edges from G.

Example 3.1. Calculating the crossing number of $P(8, 3)$.

Solution. By deleting three edges from $P(8, 3)$ we get two embeddings d_{3-1-1} and d_{3-2-1} (see Fig. 2). d_{3-1-1} is got by deleting edges $(0, 7)$, $(2, \underline{2})$, $(\underline{0}, \underline{5})$ from $P(8, 3)$. d_{3-2-1} is got by deleting edges $(0, 7)$, $(2, 3)$, $(4, \underline{4})$ from $P(8, 3)$.

Table 3 shows the steps of calculating $cr(P(8, 3))$ by Algorithm CCN, where $(w, x)(y, z)$ means that the algorithm puts back the edge $e_1 = (w, x)$ and e_1 crosses $e_2 = (y, z)$; $\nu^*_{(w,x)}$ represents the minimum crossing increment induced by putting e_1 back to d'_{j-1}; and cr_1 represents the current minimum number of crossings. By Table 3, $cr(P(8, 3)) = cr_1 = 4$.

Table 3
The steps of calculating $cr(P(8,3))$ by Algorithm CCN.

Embedding	Step	Add edges and present cr_1, cr_2
	0	$cr_1 = \infty$
d_{3-1-1}	1	$cr_2 = \nu^*_{(0,7)} + \nu^*_{(2,\underline{2})} + \nu^*_{(\underline{0},\underline{5})} = 1 + 1 + 2 = 4$
	2	$(0,7)(\underline{1},\underline{6})$, $cr_2 = 1 + \nu^*_{(2,\underline{2})} + \nu^*_{(\underline{0},\underline{5})} = 1 + 1 + 2 = 4$
	3	$(0,7)(\underline{1},\underline{6})$, $(2,\underline{2})(4,\underline{4})$, $cr_2 = 2 + \nu^*_{(\underline{0},\underline{5})} = 2 + 2 = 4$
	4	$(0,7)(\underline{1},\underline{6})$, $(2,\underline{2})(4,\underline{4})$, $(\underline{0},\underline{5})((\underline{1},\underline{6})(7,6))$, $cr_1 = 4$
d_{3-2-1}	5	$cr_2 = \nu^*_{(0,7)} + \nu^*_{(2,3)} + \nu^*_{(4,\underline{4})} = 1 + 1 + 1 = 3$
	6	$(0,7)(\underline{1},\underline{6})$, $cr_2 = 1 + \nu^*_{(2,3)} + \nu^*_{(4,\underline{4})} = 1 + 1 + 2 = 4$
	7	$(0,7)(\underline{2},\underline{5})$, $cr_2 = 1 + \nu^*_{(2,3)} + \nu^*_{(4,\underline{4})} = 1 + 2 + 1 = 4$
	8	$(2,3)(\underline{5},\underline{0})$, $cr_2 = 1 + \nu^*_{(0,7)} + \nu^*_{(4,\underline{4})} = 1 + 1 + 1 = 3$
	9	$(2,3)(\underline{5},\underline{0})$, $(0,7)(\underline{1},\underline{6})$, $cr_2 = 2 + \nu^*_{(4,\underline{4})} = 2 + 2 = 4$
	10	$(2,3)(\underline{5},\underline{0})$, $(4,\underline{4})(6,\underline{6})$, $cr_2 = 2 + \nu^*_{(0,7)} = 2 + 2 = 4$
	11	$(4,\underline{4})(6,\underline{6})$, $cr_2 = 1 + \nu^*_{(0,7)} + \nu^*_{(2,3)} = 1 + 1 + 1 = 3$
	12	$(4,\underline{4})(6,\underline{6})$, $(0,7)(\underline{2},\underline{5})$, $cr_2 = 2 + \nu^*_{(2,3)} = 2 + 2 = 4$
	13	$(4,\underline{4})(6,\underline{6})$, $(2,3)(\underline{5},\underline{0})$, $cr_2 = 2 + \nu^*_{(0,7)} = 2 + 2 = 4$

4. Conclusion

In this paper, we examine all the crossing numbers of $P(n,k)$ for n up to 16. The values of $cr(P(n,k))$ are shown in Table 2.

By Lemma 2.5, we have $cr(P(4h+2,2h)) \le 2h+1$, $h \ge 3$; and $cr(P(4h+2,4)) \le 2h+2$, $h \ge 3$. Furthermore, we have the following

Conjecture 4.1. (1) *$cr(P(4h+2,2h)) = 2h+1$ for $h \ge 3$,*
(2) *$cr(P(4h+2,4)) = 2h+2$ for $h \ge 3$.*

By Table 2, the conjecture holds for $h = 3$.

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